

# SOLVING $\bar{\partial}_b$ ON HYPERBOLIC LAMINATIONS

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ABSTRACT. Let  $X$  denote a compact set which is laminated by Riemann surfaces. We assume that  $X$  carries a positive CR line bundle  $L \rightarrow X$ . The main result of the paper is that there exists a positive integer  $s$  so that if  $v$  is any continuous  $(0, 1)$  form with coefficients in  $L^{\otimes s}$  there exists a continuous section  $u$  of  $L^{\otimes s}$  solving the equation  $\bar{\partial}_b u = v$ .

## 1. INTRODUCTION

The Cauchy-Riemann equations or the  $\bar{\partial}$  equation are among the most important tools in complex analysis. This is true in one complex dimension as well as in several complex variables. On CR manifolds one has similarly the tangential Cauchy-Riemann equations. In this paper we will study the special case of compact CR manifolds which are Leviflat and foliated by Riemann surfaces. In this case the tangential Cauchy-Riemann equations reduce to the  $\bar{\partial}$  equation on the individual leaves. Since the manifolds are compact one cannot expect to solve the  $\bar{\partial}$ -equation for  $(0, 1)$ -forms in general, and the natural thing is to consider sections of positive line bundles over the manifolds. Then of course, by classical theory, we may solve  $\bar{\partial}$  on each individual leaf - the difficulty is to obtain transversal regularity, *i.e.*, that the solutions vary nicely when you compare nearby leaves. Our main result is:

**Theorem 1.1.** *Let  $X$  be a compact hyperbolic Riemann surface lamination with a CR line bundle  $L \rightarrow X$ , and assume that  $L$  is equipped with a positive metric  $\sigma$ . Then there exists an integer  $s \in \mathbb{N}$  such that for any continuous  $(0, 1)$ -form  $v$  with coefficients in  $L^{\otimes s}$ , there exists a continuous section  $u$  of  $L^{\otimes s}$  solving  $\bar{\partial}_b u = v$ .*

In Xiaoi Chai [4], the analogous result was proved in general for the equation  $du/dx = v$  for arbitrary foliations by real curves.

We will prove a stronger version of this theorem, for the special case of a suspension over a compact Riemann surface, and as an application we will prove the following:

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*Date:* August 12, 2011.

<sup>1</sup>The first author is supported by an NSF grant DMS-1006294.

Keywords: Cauchy-Riemann equations, Levi flat manifolds, Kobayashi hyperbolicity, Riemann surface foliations.

2010 AMS classification. Primary: 32V20, 32W10; Secondary: 32F45

**Theorem 1.2.** *There exists a  $C^1$ -smooth hyperbolic minimal Riemann surface lamination in  $\mathbb{P}^5$  with uncountably many extremal closed laminated currents which are mutually disjoint.*

The theorem is proved by solving  $\bar{\partial}_b$  with *smooth* transverse regularity for suspensions (see (5.1)), and thereby obtaining an embedding theorem à la Kodaira, applied to a suspension considered in [12]. Related to such an embedding result, Ghys [9] and Deroin [5] have shown that meromorphic functions and projective maps separate points on these laminations (see also Gromov [10]).

Theorem 1.2 is in strong contrast to the situation in  $\mathbb{P}^2$  where any such lamination supports a unique normalized  $\partial\bar{\partial}$ -closed laminated current [11]. We will discuss this in Section 6.

In a sequel to this paper, we will discuss further applications, and also non hyperbolic laminations, as well as laminations without positive CR bundles.

We next describe the plan of the paper. In Section 2 we discuss the Kobayashi metric  $K_x$  on leaves of a hyperbolic foliation  $X$ , and we give a new proof of the upper semi continuity of  $K_x$ . Then in Section 3 we show some useful facts for the unit disc. In Section 4 we discuss families of positive line bundles over the unit disc and prove continuity estimates for  $\bar{\partial}$  with values in these line bundles. The line bundles are trivial but the metric varies quite strongly. In Section 5 we will prove the main theorem for a specific example: the case of a suspension over a compact Riemann surface. The proof will give the main ideas for the general case, it will reveal the further need for the results in section 2, but the situation being somewhat simpler than the general case, we will not need section 4. We also obtain stronger transverse regularity in this special case. In Section 6 we prove Theorem 1.2 via an embedding result à la Kodaira. Finally we prove the main theorem solving  $\bar{\partial}_b$  in Section 7.

The Cauchy-Riemann equations have been discussed in the Levi flat Riemann surface case before by several authors, see [3], [13], [14], [16] and references therein.

## 2. HYPERBOLIC LAMINATIONS AND THE KOBAYASHI METRIC

We first define what we mean by a Riemann surface lamination. Let  $X$  be a topological space with an open cover  $\{U_\alpha\}_{\alpha \in A}$ . We assume that for each  $\alpha$  there is a homomorphism  $\phi_\alpha : U_\alpha \rightarrow \mathbb{D}(z_\alpha) \times T_\alpha(t_\alpha)$  where  $\mathbb{D}$  is the open unit disc in  $\mathbb{C}$  and  $T_\alpha$  is a metrizable topological space. Moreover the maps  $\phi_{\beta\alpha} := \phi_\beta \circ \phi_\alpha^{-1}$  have locally the form  $\phi_{\beta\alpha}(z_\alpha, t_\alpha) = (z_{\beta\alpha}(z_\alpha, t_\alpha), t_{\beta\alpha}(t_\alpha))$  where the function  $z_\beta$  is holomorphic as a function of  $z_\alpha$  for fixed  $t_\alpha$ .

The sets  $U_\alpha$  are called flow boxes. The sets  $\mathcal{L}_{\alpha, t_\alpha} := \phi_\alpha^{-1}(\mathbb{D} \times \{t_\alpha\})$  are called plaques and are homeomorphic to the unit disc. A nonempty subset  $\mathcal{L} \subset X$  is called a leaf (of the lamination) if whenever  $x \in \mathcal{L} \cap U_\alpha$  for some

$\alpha$  then  $\mathcal{L}$  contains the plaque in  $U_\alpha$  containing  $x$  and moreover  $\mathcal{L}$  is minimal with respect to this condition. The set  $X$  is then a disjoint union of leaves and for every  $x$ , the leaf through  $x$ ,  $\mathcal{L}_x$ , consists of all points in  $X$  which can be joined to  $x$  with a curve which is locally contained in a plaque. A basis for a topology on a leaf  $\mathcal{L}$  is given by proclaiming that each plaque in  $\mathcal{L}$  is an open set, and that each set  $U \cap \mathcal{L}$  is open, where  $U$  is an open subset of  $X$ . Then each leaf is a Hausdorff topological space, and each leaf has a natural structure of a Riemann surface inherited from the maps  $\phi_\alpha$ . We say that a Riemann surface lamination is hyperbolic if each leaf is hyperbolic, *i.e.*, it is universally covered by the unit disk.

Let  $L \rightarrow X$  be a continuous complex line bundle. We will say that  $L$  is a complex line bundle on  $X$  if it is defined by transition functions  $f_{\alpha\beta}$  on  $U_\alpha \cap U_\beta$ , where  $f_{\alpha\beta}$  is holomorphic along plaques. By a smooth section of  $L$  we will mean a continuous section which is smooth along the leaves. A weight  $\sigma$  on  $L$  will be a family of continuous functions  $\sigma_\alpha$  on  $U_\alpha$ , smooth along the plaques, with  $\sigma_\alpha = \sigma_\beta + 2 \cdot \log |f_{\alpha\beta}|$  on  $U_\alpha \cap U_\beta$ . We also assume that all partial derivatives of each  $\sigma_\alpha$  vary continuously between leaves. The weight  $\sigma$  is said to be positive if each  $\sigma$  is strictly subharmonic along the leaves.

For a Riemann surface lamination the notions of the tangent- and co-tangent bundle only have meaning along the leaves. Considering these however, we have a natural definition of  $(0, 1)$ -forms with coefficients in  $L$ , and also the  $\bar{\partial}$ -operator acting on sections along the leaves, denoted by  $\bar{\partial}_b$ . A  $(0, 1)$ -form is said to be smooth if, in local coordinates, it is continuous and smooth along plaques. Note that if  $v_t(z)$ ,  $t \in \mathbb{T}$ , is a family of continuous  $(0, 1)$ -forms on the disk  $\mathbb{D}$ , continuous in the parameter  $t$ , and if  $u_t$  is a family of  $L^2$ -functions solving  $\bar{\partial}u_t = v_t$  in the weak sense, and also  $\|u_t - u_{t_0}\|_{L^2(\mathbb{D})} \rightarrow 0$  as  $t \rightarrow t_0$  for all  $t_0 \in \mathbb{T}$ , then  $u_t(z)$  is continuous in both variables. This follows from the facts that the family of solutions given by the Cauchy integral has this property, that weakly holomorphic functions are holomorphic, and the  $L^2$  to sup-norm estimate.

Let  $X$  be a Riemann surface lamination, and assume that  $X$  is equipped with a hermitian metric  $\|\cdot\|_X$  along leaves, varying continuously also between leaves. For a point  $x \in X$  let  $\mathcal{F}_x$  denote the family of holomorphic maps  $f : \mathbb{D} \rightarrow \mathcal{L}_x$  with  $f(0) = x$ . For a map  $f \in \mathcal{F}_x$  and  $\zeta \in \mathbb{D}$  we let  $f'(\zeta)$  denote the tangent vector  $f_*(\zeta)(\partial/\partial\zeta)$ . The Kobayashi metric  $K = K_X$  at a point  $x \in X$  is defined by

$$K_x(x) := [\sup_{f \in \mathcal{F}_x} \{\|(f'(0))\|_X\}]^{-1}.$$

Our approach to solving  $\bar{\partial}_b$  on hyperbolic laminations will be to lift the problem to line bundles over the unit disk (the universal covers of the leaves), solve the  $\bar{\partial}$ -equations there according to a certain procedure, and then push the solutions back down. It will therefore be important to understand how

the leaves distribute, and, moreover, that the universal covering maps vary regularly when we pass between leaves. We will need the following result regarding the Kobayashi metric  $K_x$ :

**Theorem 2.1.** *Let  $X$  be a compact Riemann surface lamination, and assume that all leaves in  $X$  are hyperbolic. Then  $K_x$  is a continuous function on  $X$ . Moreover, if  $x_j$  is a sequence of points in  $X$  converging to a point  $x_0 \in X$ ,  $v_j$  is a sequence of tangent vectors at the points  $x_j$  converging to a nonzero tangent vector  $v_0$  at  $x_0$ , and if  $f_j : \mathbb{D} \rightarrow \mathcal{L}_{x_j}$  are the universal covering maps with  $f_j(0) = x_j$ ,  $f'_j(0) = \lambda_j \cdot v_j$ ,  $\lambda_j > 0$ , then the sequence  $f_j$  converges uniformly on compacts to the universal covering map  $f_0 : \mathbb{D} \rightarrow \mathcal{L}_x$  with  $f(0) = x$  and  $f'(0) = \lambda_0 \cdot v_0$ ,  $\lambda_0 > 0$ .*

This result was proved by Candel [2] (see also [8],[17]). Recently, Dinh, Nguyen and Sibony [6] proved that  $K_x$  is actually Hölder continuous. Lower semi-continuity is proved following Brody [1]: we first obtain a strictly positive lower bound  $c > 0$  for  $K$ ; otherwise we could produce a non-degenerate image of  $\mathbb{C}$  in a leaf  $\mathcal{L}$ . Knowing this we have that any holomorphic map  $f : \mathbb{D} \rightarrow \mathcal{L}$  satisfies  $|f'(\zeta)| \leq c \cdot (1/(1 - |\zeta|^2))$ . Hence  $\mathcal{F} := \{Hol(\mathbb{D}, L_x) : x \in X\}$  is sequentially compact, and so  $K$  is lower semi-continuous. Assume for a moment that we also know that  $K$  is upper semicontinuous, and note the following: if  $x \in X$  is a point, and  $f : \mathbb{D} \rightarrow \mathcal{L}_x$  is holomorphic,  $f(0) = x$ , then  $f$  is a universal covering map if and only if  $|f'(0)| = K_x^{-1}$ ; this follows from the Schwarz lemma and the fact that such map can be factored through the universal covering map. Hence any convergent sequence of universal covering maps is a universal covering map, and so we easily obtain the last claim of the theorem.

We will give a new proof of the upper semi-continuity of  $K_X$ . In the case where  $X$  is a complex manifold, the upper semi-continuity of  $K_X$  is a well known theorem of Royden [15]. The crucial point in his approach is to prove that if  $f : \mathbb{D} \rightarrow X$  is an embedding and if  $r < 1$ , then  $f(\overline{\mathbb{D}}_r)$  admits a Stein neighborhood in  $X$ , so the strategy is not immediately applicable in the case of laminations.

**Theorem 2.2.** *Let  $X$  be any Riemann surface lamination. Then the  $K_X$  is upper semi-continuous on  $X$ .*

*Proof.* Let  $x \in X$  and let  $f : \mathbb{D} \rightarrow \mathcal{L}_x$  be a holomorphic map with  $f(0) = x$  and  $f'(0) \neq 0$ . Let  $x_j$  be a sequence of points in  $X$  converging to  $x$ . We will show that for any  $0 < r < 1$ , the map  $f_r := f|_{\mathbb{D}_r}$  is the uniform limit of maps  $f_j : \mathbb{D}_r \rightarrow \mathcal{L}_{x_j}$ .

Let  $Z \subset \mathbb{D}_r$  denote the singular locus of  $f$ , i.e., the finite set of points where  $f'$  vanishes. We will cover  $\overline{\mathbb{D}}_r$  by a suitable finite increasing sequence  $A_j$  of closed topological disks. Each  $A_j$  is obtained by defining  $A_j := A_{j-1} \cup B_j$  where  $B_j$  is a closed topological disk for  $j \geq 2$ , and  $A_1 = B_1$ . We want that

- 1)  $C_j := \overline{B_{j+1}} \cap A_j \neq \emptyset$ ,
- 2)  $(\overline{A_{j-1}} \setminus \overline{B_j}) \cap (\overline{B_j} \setminus \overline{A_{j-1}}) = \emptyset$ ,
- 3)  $Z \cap C_j = \emptyset$ , and
- 4)  $f(B_j)$  is contained in a coordinate chart  $U_j \subset X$  for each  $j$ .

We make sure that each  $C_j$  has an open neighborhood  $\tilde{C}_j$  in  $\mathbb{D}$  such that

- 5)  $f|_{\tilde{C}_j}$  is injective.

By choosing  $A_1$  small enough there is an open neighborhood  $W_1$  of  $A_1$  in  $\mathbb{D}$  such that  $f : \overline{W_1} \rightarrow \mathcal{L}_x$  is the uniform limit of a sequence  $f_j : \overline{W_1} \rightarrow \mathcal{L}_{x_j}$ ; simply lift  $f$  within a flowbox. We will proceed by induction.

Assume that we have found an open neighborhood  $W_k$  of  $A_k$ ,  $k \geq 1$ , such that  $f : \overline{W_k} \rightarrow \mathcal{L}_x$  is the uniform limit of a sequence  $f_j : \overline{W_k} \rightarrow \mathcal{L}_{x_j}$ . Choose an open neighborhood  $\hat{B}_{k+1}$  of  $B_{k+1}$  such that  $f(cl(\hat{B}_{k+1}))$  is contained in a coordinate chart. By possibly having to choose a smaller  $\tilde{C}_k$  we may assume that  $\tilde{C}_k \subset \subset W_k \cap \hat{B}_{k+1}$ .

According to [7], Theorem 4.1., there exist open neighborhoods  $A'_k$ ,  $B'_{k+1}$  and  $C'_k$  of  $A_k$ ,  $B_{k+1}$  and  $C_k$  respectively,  $C'_k \subset A'_k \cap B'_{k+1} \subset \tilde{C}_k$ , such that if  $\gamma : \tilde{C}_k \rightarrow \mathbb{D}$  is a holomorphic map sufficiently close to the identity, then there exist injective holomorphic maps  $\alpha : A'_k \rightarrow \mathbb{D}$  and  $\beta : B'_{k+1} \rightarrow \mathbb{D}$  such that  $\gamma = \beta \circ \alpha^{-1}$  on  $C'_k$ . Moreover,  $\alpha$  and  $\beta$  can be assumed uniformly close to the identity, depending on  $\gamma$ .

Fix a flow box containing  $f(cl(\hat{B}_{k+1}))$ , and let  $g_j : \hat{B}_{k+1} \rightarrow \mathcal{L}_{x_j}$  be the sequence of maps obtained by lifting  $f : \hat{B}_{k+1}$  to the leaf  $\mathcal{L}_{x_j}$ . Then  $\gamma_j := g_j^{-1} \circ f_j \rightarrow id$  uniformly on  $\tilde{C}_j$  as  $j \rightarrow \infty$ . Let  $\alpha_j, \beta_j$  be a sequence of splittings as alluded to above. Then if we choose a small enough open neighborhood  $W_{k+1}$  of  $A_{k+1}$  we have that the map  $\tilde{f}_j$  defined as  $\tilde{f}_j := f_j \circ \alpha_j$  near  $A_k$  and  $\tilde{f}_j = g_j \circ \beta_j$  near  $B_{k+1}$  are well defined on  $\overline{W}_{k+1}$  and converges uniformly to the map  $f$  as  $k \rightarrow \infty$ .

□

### 3. PREPARATIONS FOR THE ANALYSIS ON FAMILIES OF UNIT DISCS

In this section we discuss decompositions of the unit disc which reflects the way the disc covers leaves of hyperbolic laminations, and we also estimate the deck transformations. This will be used in the next section to investigate the  $\bar{\partial}$ - equation for data pulled back from the lamination.

**Definition 3.1.** Throughout this paper we let  $\psi(\zeta)$  denote the function  $\psi(\zeta) = \log(1 - |\zeta|^2)$  defined on the unit disk  $\mathbb{D}$  in  $\mathbb{C}$ .

Note that the Poincaré metric on the disk is given by  $P(\zeta) = e^{-\psi(\zeta)}|d\zeta|$ , and recall that the Poincaré distance  $d_P(0, a)$  between the origin and a point  $a \in \mathbb{D}$  is given by  $d_P(0, a) = \frac{1}{2} \log[(1 + |a|)/(1 - |a|)]$ .

**Definition 3.2.** We let  $A_n := \{\zeta \in \mathbb{D} : 1 - (\frac{1}{2})^n \leq |\zeta| < 1 - (\frac{1}{2})^{n+1}\}$ . We let  $\mathbb{D}(n) = \{\zeta \in \mathbb{D} : |\zeta| \leq 1 - (\frac{1}{2})^{n+1}\}$ .

Note that for all  $a \in A_n$  we have that

- i)  $(\frac{1}{2})^{n+1} \leq 1 - |a|^2 \leq (\frac{1}{2})^{n-1}$ ,
- ii)  $2^{n-1} \leq e^{-\psi(a)} \leq 2^{n+1}$ , and
- iii)  $d_P(0, a) \leq n + 2$ .

**Lemma 3.3.** Let  $\phi \in \text{Aut}_{hol} \mathbb{D}$  with  $\phi(0) \in A_n$ . Then  $e^{-\psi(\phi(\zeta))} \leq 2^{n+3} e^{-\psi(\zeta)}$  for all  $\zeta \in \mathbb{D}$ .

*Proof.* Write  $\phi(\zeta) = e^{i\beta}(\zeta - a)/(1 - \bar{a}\zeta)$  with  $a \in A_n$ . Then  $\phi'(\zeta) = e^{i\beta} \cdot (1 - a\bar{a})/(1 - \bar{a} \cdot \zeta)^2$ , and so  $|\phi'(\zeta)| \geq \frac{1}{4}(1 - |a|^2)$  for all  $\zeta \in \mathbb{D}$ . By the Schwarz-Pick Lemma we also have that  $|\phi'(\zeta)| = (1 - |\phi(\zeta)|^2)/(1 - |\zeta|^2)$ , so we get that  $1/(1 - |\phi(\zeta)|^2) \leq 4/[(1 - |a|^2)(1 - |\zeta|^2)]$ .  $\square$

**Lemma 3.4.** Let  $\phi \in \text{Aut}_{hol} \mathbb{D}$  with  $\phi(0) \in A_n$ . Then

$$\mathbb{D}_{(\frac{1}{2})^{n+k+3}}(\phi(0)) \subset \phi(\mathbb{D}_{(\frac{1}{2})^k}).$$

*Proof.* From the previous proof we have that  $|\phi'(\zeta)| \geq (\frac{1}{2})^{n+3}$ .  $\square$

**Lemma 3.5.** Let  $\phi \in \text{Aut}_{hol} \mathbb{D}$  with  $\phi(0) \in A_n$ . Then

$$|\phi'(\zeta)| \leq 2^{n+2}$$

for all  $\zeta \in \mathbb{D}$ .

*Proof.* We have  $|\phi'(\zeta)| = (1 - |a|^2)/|(1 - \bar{a}\zeta)|^2 \leq (1 + |a|)/(1 - |a|)$ .  $\square$

**Lemma 3.6.** Let  $\phi \in \text{Aut}_{hol} \mathbb{D}$  with  $\phi(0) \in A_n$ . Then

$$|\phi'(\zeta)| \leq (1 - r)^{-2} (\frac{1}{2})^{n-1}$$

for  $\zeta \in \mathbb{D}_r$ .

*Proof.* We have that  $|\phi'(\zeta)| = |(1 - |a|^2)|/|(1 - \bar{a}\zeta)|^2$  for all  $\zeta \in \mathbb{D}$ .  $\square$

**Lemma 3.7.** Let  $\phi \in \text{Aut}_{hol} \mathbb{D}$  with  $\phi(0) \in A_n$ . For any  $0 < r < 1$  we have that

$$|\phi(\zeta)| \geq 1 - (\frac{1}{2})^n \cdot [1 + \frac{2r}{(1 - r)^2}],$$

for all  $\zeta \in \mathbb{D}_r$ .

*Proof.* This follows from Lemma 3.6 and the mean value theorem.  $\square$

**Lemma 3.8.** Let  $r > 0$ . There exists a constant  $c > 0$  such that

$$e^{-\psi(\phi(\zeta))} \geq c \cdot 2^n,$$

for  $\phi \in \text{Aut}_{hol} \mathbb{D}$  with  $\phi(0) \in A_n$ , for all  $\zeta \in \mathbb{D}_r$  and all  $n \in \mathbb{N}$ .

*Proof.* Use the previous lemma.  $\square$

Let  $S$  denote the strip  $\{\zeta \in \mathbb{C} : 0 < \operatorname{Re}(\zeta) < 1\}$ . For  $k = 0, 1, \dots, 2^5 - 2$  and  $l \in \mathbb{Z}$  let  $S_{k,l}$  denote the rectangle

$$S_{k,l} := \{x+iy \in S : k \cdot (\frac{1}{2})^5 < x < (k+2) \cdot (\frac{1}{2})^5 \text{ and } l \cdot (\frac{1}{2})^8 < y < (l+2) \cdot (\frac{1}{2})^8\}.$$

Choose a partition of unity  $\alpha_{k,l}$  with respect to the cover  $\{S_{k,l}\}$  of  $S$  which is translation invariant in the  $y$ -direction, *i.e.*,  $\alpha_{k,l+j}(x, y) = \alpha_{k,l}(x, y - j \cdot (\frac{1}{2})^8)$ . Let  $c_p > 0$  be a constant such that  $\|\alpha_{k,l}\|_{C^1(S_{k,l})} \leq c_p$  for all  $k, l$ .

For  $n = 1, 2, \dots$  let  $f_n : S \rightarrow A_n$  denote the map

$$f_n(x, y) = (1 - (\frac{1}{2})^n + x \cdot (\frac{1}{2})^{n+1})e^{2\pi i y (\frac{1}{2})^{n+1}}.$$

For  $n = 1, 2, \dots$  let  $\tilde{S}_{k,l,n} = f_n(S_{k,l})$  for  $k = 0, \dots, 2^5 - 2$  and  $l = 0, 1, \dots, 2^{n+9} - 1$ , and let  $\tilde{\alpha}_{k,l,n}$  denote the function  $\tilde{\alpha}_{k,l,n} = \alpha_{k,l} \circ f_n^{-1}$ . Then  $\{\tilde{\alpha}_{k,l,n}\}$  is a partition of unity with respect to the cover  $\{\tilde{S}_{k,l,n}\}$  of  $A_n^\circ$ . Note that

1. any point  $\zeta \in A_n^\circ$  is contained in at most four  $\tilde{S}_{k,l,n}$ 's,
2. if  $a \in \tilde{S}_{k,l,n}$  and  $\phi \in \operatorname{Aut}_{hol} \mathbb{D}$  satisfies  $\phi(0) = a$ , then  $\tilde{S}_{k,l,n} \subset \phi(\mathbb{D}_{\frac{1}{2}})$  (Lemma 3.4),
3. there exists a constant  $\tilde{c}_p > 0$  such that  $\|\tilde{\alpha}_{k,l,n}\|_{C^1(\tilde{S}_{k,l,n})} \leq \tilde{c}_p \cdot 2^n$  for all  $k, l, n$ .

It follows from Lemma 3.4, Lemma 3.6, and 3., that

4. there exists a constant  $\tilde{c} > 0$  such that if  $a \in \tilde{S}_{k,l,n}$  and  $\phi \in \operatorname{Aut}_{hol} \mathbb{D}$  satisfies  $\phi(0) = a$ , then  $\|\tilde{\alpha}_{k,l,n} \circ \phi\|_{C^1(\phi^{-1}(\tilde{S}_{k,l,n}))} \leq \tilde{c}$  for all  $k, l, n$ .

Let  $\chi(x)$  be a decreasing function which is one on the interval  $(0, \frac{1}{4})$  and which is zero on  $(\frac{3}{4}, 1)$ . Let  $\tilde{\chi}_n := \chi \circ f_n^{-1}$ . We may assume that

5. if  $a \in \tilde{S}_{k,l,n}$  and  $\phi \in \operatorname{Aut}_{hol} \mathbb{D}$  satisfies  $\phi(0) = a$ , then  $\|\tilde{\chi}_n \circ \phi\|_{C^1(\phi^{-1}(\tilde{S}_{k,l,n}))} \leq \tilde{c}$  for all  $k, l, n$ .

**Lemma 3.9.** *For any  $r > 0$  there exists a constant  $c > 0$  such that if  $E \subset A_n, n \in \mathbb{N}$ , is a set of points with  $d_P(e_1, e_2) \geq r$  for all  $e_1, e_2 \in E$  with  $e_1 \neq e_2$ , then  $\sharp(E) \leq c \cdot 2^n$ .*

*Proof.* Fix  $k \in \mathbb{N}$  such that the Poincaré radius of the disk  $\mathbb{D}_{(\frac{1}{2})^k}$  is less than  $\frac{r}{2}$ . By Lemma 3.4 we have that if  $\phi \in \operatorname{Aut}_{hol} \mathbb{D}$  with  $\phi(0) \in A_n$  then  $D_{(\frac{1}{2})^{n+k+3}}(\phi(0)) \subset \phi(\mathbb{D}_{(\frac{1}{2})^k})$ . Copy the construction of the cubes  $S_{k,l}$  as above, but with sides of length  $(\frac{1}{2})^{k+3}$  and  $(\frac{1}{2})^{k+6}$  respectively. Then the corresponding cubes  $\tilde{S}_{k,l}$  have diameters less than  $(\frac{1}{2})^{n+k+3}$ , and a number  $2^{k+4-2} \cdot (2^{n+k+8} - 1)$  of cubes is needed to cover  $A_n$ . □

## 4. FAMILIES OF LINE BUNDLES OVER THE DISK

Our approach to solve  $\bar{\partial}_b$  on lamination will be to solve  $\bar{\partial}$  for sections of positive line bundles over  $\mathbb{D}$ . Let  $L \rightarrow \mathbb{D}$  be a line bundle with a positive metric  $\sigma$ . Since any line bundle over  $\mathbb{D}$  is trivial, we may solve  $\bar{\partial}$  using Hörmander: assume that  $dd^c\sigma \geq c \cdot dV$ , and let  $v \in L^2_{(0,1)}(L, \sigma)$ . Then there exists  $u \in L^2(L, \sigma)$  with  $\bar{\partial}u = v$  and

$$\int_{\mathbb{D}} |u|^2 e^{-\sigma} dV \leq \frac{1}{c} \int_{\mathbb{D}} |v|^2 e^{-\sigma} dV.$$

We need to study how these (canonical) solutions vary for certain families of line bundles over  $\mathbb{D}$ .

Given an open set  $U \subset \mathbb{C}$  we let  $\|\cdot\|_{U,1}$  denote the norm

$$\|g\|_{U,1} := \sup_{\zeta \in U, s+t \leq 1} \{ |(\partial^{s+t}g)/(\partial x^s \partial y^t)(\zeta)| \},$$

defined for each  $g \in \mathcal{C}^1(U)$ . Note that if  $\sigma_1$  and  $\sigma_2$  are metrics on a line bundle  $L \rightarrow U$ , then the difference  $\sigma_1 - \sigma_2$  is a function on  $U$ .

Let  $\{U_j\}_{j=1}^\infty$  be a locally finite cover of the disk  $\mathbb{D}$ , and let  $\mathbb{T}$  be a topological space. We shall consider families of line bundles over  $\mathbb{D}$  parametrized by  $\mathbb{T}$ . A line bundle  $L_t$  is given by a collection of transition functions  $f_{t,i,j} \in \mathcal{O}(U_{ij})$ , and a metric  $\sigma_t$  on  $L_t$  is given by a collection  $\sigma_{t,j}$  of locally integrable functions, satisfying the compatibility condition

$$\sigma_{t,i} - \sigma_{t,j} = 2 \cdot \log |f_{t,i,j}|$$

on  $U_{ij}$ . We will assume that

1. there exists a constant  $c > 0$  such that  $dd^c\sigma_t \geq \frac{c}{(1-|\zeta|^2)^2} dV$  for all  $t \in \mathbb{T}$ ,
2. for any pair  $i, j \in \mathbb{N}$  and any  $t_0 \in \mathbb{T}$  we have that  $\|f_{t,i,j} - f_{t_0,i,j}\|_{U_{ij},1} \rightarrow 0$  as  $t \rightarrow t_0$ , and
3. for any  $j \in \mathbb{N}$  and any  $t_0 \in \mathbb{T}$  we have that  $\|\sigma_{t,j} - \sigma_{t_0,j}\|_{U_{j,1}} \rightarrow 0$  as  $t \rightarrow t_0$ .

**Remark 4.1.** By 3. it is understood that the non-smooth parts of the metrics cancel.

We may of course regard the union of the  $L_t$ -s as a bundle over  $\mathbb{D} \times \mathbb{T}$ . We denote this bundle by  $L_{\mathbb{T}}$ .

**Remark 4.2.** Note that

$$dd^c(s\psi)(\zeta) = \frac{-4s}{(1-|\zeta|^2)^2} dV,$$

so if  $c > 4s$  we may solve  $\bar{\partial}$  for sections in  $L^2(L_t, \sigma_t + s\psi)$  with estimates:  $dd^c(\sigma_t + s\psi) \geq (c - 4s)dV$ . Also, if  $\varphi \in \text{Aut}_{hol} \mathbb{D}$  then  $dd^c(\varphi^*\psi) = dd^c\psi$ , and so  $dd^c(\varphi^*\sigma_t + s\psi) \geq (c - 4s)dV$ .



The following is the main result of this section.

**Theorem 4.3.** *Let  $\{L_t, \sigma_t\}_{t \in \mathbb{T}}$  be a family of line bundles satisfying 1–3. Let  $s \in \mathbb{N}$  and assume that  $c > 4s$ . Let  $V \subset \subset \mathbb{D}$  be a domain, and let  $v_t \in \mathcal{C}_{(0,1)}(L_t, \sigma_t)$  be a continuous family of forms supported in  $V$ . For each  $t \in \mathbb{T}$  let  $u_t$  be the  $L^2(L_t, \sigma_t + s\psi)$ -minimal solution to the equation  $\bar{\partial}u_t = v_t$ . Then  $u_t$  is a continuous section of  $L_{\mathbb{T}}$ .*

We prove first some intermediate results, and then we prove the theorem at the end of the section .

**Proposition 4.4.** *Let  $\{L_t, \sigma_t\}_{t \in \mathbb{N}}$  be a family of line bundles satisfying 1–3, let  $s \in \mathbb{N}$  and assume that  $c > 4s$ . There exists a constant  $c_1 > 0$  such that the following holds:*

*For any  $t \in \mathbb{T}$  and for any section  $u \in \mathcal{O}L^2(L_t|_{A_n}, \sigma_t + s\psi)$ , define  $v := \bar{\partial}(\tilde{\chi}_n \cdot u)$  ( $v = 0$  over  $\mathbb{D} \setminus A_n$ ). Then there exists  $u_n \in \mathcal{C}^\infty L^2(L_t, \sigma_t + s\psi)$  with  $\bar{\partial}u_n = v$ , and*

$$\int_{\mathbb{D}} |u_n|^2 e^{-(\sigma_t + s\psi)} dV \leq c_1 \int_{A_n} |u|^2 e^{-(\sigma_t + s\psi)} dV.$$

*Proof.* We use the partition of unity  $\{\tilde{\alpha}_{k,l,n}\}$  with respect to  $\{\tilde{S}_{k,l,n}\}$  defined in Section 3, and we write

$$v = \bar{\partial}(\tilde{\chi}_n \cdot u) = \sum_{k,l} \tilde{\alpha}_{k,l,n} \cdot \bar{\partial}(\tilde{\chi}_n \cdot u).$$

Note that  $v_{k,l,n} := \tilde{\alpha}_{k,l,n} \cdot \bar{\partial}(\tilde{\chi}_n \cdot u)$  is  $\mathcal{C}^\infty$ -smooth on  $\mathbb{D}$  and is supported in  $\tilde{S}_{k,l,n}$ .

**Lemma 4.5.** *There exists a constant  $c_3 > 0$ , independent of  $k, l, n$ , such that the following holds:*

*There exists a section  $u_{k,l,n} \in \mathcal{C}^\infty L^2(L_t, \sigma_t + s\psi)$  with  $\bar{\partial}u_{k,l,n} = v_{k,l,n}$ , and*

$$\int_{\mathbb{D}} |u_{k,l,n}|^2 e^{-(\sigma_t + s\psi)} dV \leq c_3 \int_{\tilde{S}_{k,l,n}} |u|^2 e^{-(\sigma_t + s\psi)} dV$$

*Proof.* Let  $\varphi \in \text{Aut}_{\text{hol}} \mathbb{D}$  such that  $\varphi(0) \in \tilde{S}_{k,l,n}$  and, consequently,  $\tilde{S}_{k,l,n} \subset \varphi(\mathbb{D}_{\frac{1}{2}})$ . Let  $v_{k,l,n}^*$  denote the section  $v_{k,l,n}^* := \varphi^* v_{k,l,n}$  of the bundle  $\varphi^* L_t$ . We want to solve  $\bar{\partial}u_{k,l,n}^* = v_{k,l,n}^*$ , and then push the solution back forward using  $\varphi$ . We use the metric  $\varphi^* \sigma_t + s\psi$  (see Remark 4.2). Note that, by 4. and 5. in Section 3,  $v_{k,l,n}^* = \varphi^* \tilde{\alpha}_{k,l,n} \cdot \bar{\partial}[\varphi^*(\tilde{\chi}_n \cdot u)] = ((\tilde{\alpha}_{k,l,n} \cdot u) \circ \varphi) \cdot \bar{\partial}[\varphi^*(\tilde{\chi}_n)]$ , and so  $|v_{k,l,n}^*|^2 \leq c_4 \cdot |\varphi^* u|^2$ , where  $c_4$  is independent of  $k, l, n$  and  $t$ . We have that

$$\begin{aligned}
\int_{\mathbb{D}} |v_{k,l,n}^*|^2 e^{-(\varphi^* \sigma_t + s\psi)} dV &\leq c_4 \int_{\varphi^{-1}(\tilde{S}_{k,l,n})} |\varphi^* u|^2 e^{-(\varphi^* \sigma_t + s\psi)} dV \\
&= c_4 \int_{\tilde{S}_{k,l,n}} \varphi_* [|\varphi^* u|^2 e^{-(\varphi^* \sigma_t + s\psi)}] dV \\
&= c_4 \int_{\tilde{S}_{k,l,n}} |u|^2 e^{-\sigma_t} \cdot e^{-2\psi} \cdot e^{-(s-2)(\psi \circ \varphi^{-1})} dV \\
&\leq c_5 \int_{\tilde{S}_{k,l,n}} |u|^2 e^{-\sigma_t} \cdot e^{-2\psi} dV \\
&\leq c_6 \left(\left(\frac{1}{2}\right)^n\right)^{s-2} \int_{\tilde{S}_{k,l,n}} |u|^2 e^{-\sigma_t - s\psi} dV.
\end{aligned}$$

By Hörmander there exists a section  $u_{k,l,n}^*$  solving  $\bar{\partial} u_{k,l,n}^* = v_{k,l,n}^*$  with

$$\int_{\mathbb{D}} |u_{k,l,n}^*|^2 e^{-(\varphi^* \sigma_t + s\psi)} dV \leq (c - 4s)^{-1} \cdot c_6 \left(\left(\frac{1}{2}\right)^n\right)^{s-2} \int_{\tilde{S}_{k,l,n}} |u|^2 e^{-\sigma_t - s\psi} dV.$$

Now let  $u_{k,l,n} := \varphi_* u_{k,l,n}^*$ . We get that

$$\begin{aligned}
\int_{\mathbb{D}} |u_{k,l,n}|^2 e^{-(\sigma_t + s\psi)} dV &= \int_{\mathbb{D}} \varphi_* [|u_{k,l,n}^*|^2 e^{-(\varphi^* \sigma_t + s\psi)}] dV \\
&= \int_{\mathbb{D}} |u_{k,l,n}^*|^2 e^{-\varphi^* \sigma_t} \cdot e^{-2\psi} \cdot e^{-(s-2)(\psi \circ \varphi)} dV \\
&\leq (2^{n+3})^{s-2} \int_{\mathbb{D}} |u_{k,l,n}^*|^2 e^{-\varphi^* \sigma_t} \cdot e^{-2\psi} \cdot e^{-(s-2)\psi} dV \\
&\leq (c - 4s)^{-1} \cdot c_6 \cdot 2^{3(s-2)} \int_{\tilde{S}_{k,l,n}} |u|^2 e^{-\sigma_t - s\psi} dV,
\end{aligned}$$

where in the first inequality we used Lemma 3.3.  $\square$

By Lemma 4.5 there exists for each pair  $k, l$  a section  $u_{k,l,n}$  solving  $\bar{\partial} u_{k,l,n} = \tilde{\alpha}_{k,l,n} \cdot \bar{\partial}(\tilde{\chi}_n \cdot u)$ , with

$$\int_{\mathbb{D}} |u_{k,l,n}|^2 e^{-(\sigma_t + s\psi)} dV \leq c_3 \int_{\tilde{S}_{k,l,n}} |u|^2 e^{-(\sigma_t + s\psi)} dV.$$

Define  $u_n := \sum_{k,l} u_{k,l,n}$ . Since any point  $\zeta \in A_n$  intersects at most four  $\tilde{S}_{k,l,n}$ -s we get that

$$\int_{\mathbb{D}} \int |u_n|^2 e^{-(\sigma_t + s\psi)} dV \leq 4 \cdot c_3 \int_{A_n} \int |u|^2 e^{-(\sigma_t + s\psi)} dV.$$

□

**Corollary 4.6.** *There exists a constant  $c_2$  such that the following holds. Let  $U \subset \subset \mathbb{D}$  and choose  $N \in \mathbb{N}$  such that  $A_n \cap \overline{U} = \emptyset$  for all  $n \geq N$ . Let  $v \in L^2_{(0,1)}(L_t, \sigma_t + s\psi)$  with  $v$  supported in  $U$ . Let  $u_n$  be the  $L^2(L_t|_{\mathbb{D}(n)}, \sigma_t + s\psi)$ -minimal solution to  $\bar{\partial}u_n = v$ , and let  $u$  be the  $L^2(L_t, \sigma_t + s\psi)$ -minimal solution to  $\bar{\partial}u = v$ . We extend  $u_n$  to  $\mathbb{D}$  by setting  $u_n = 0$  outside  $\mathbb{D}(n)$ . Then*

$$\|u_n - u\|_{L^2(L_t, \sigma_t + s\psi)} \leq c_2 \cdot \|u_n\|_{L^2(L_t|_{A_n}, \sigma_t + s\psi)}.$$

*Proof.* We let  $\chi_n$  denote  $\tilde{\chi}_n$  extended to be 1 on  $\mathbb{D}(n-1)$ . Let  $\tilde{u}_n := \chi_n \cdot u_n$  and let  $\tilde{u}'_n = \tilde{\chi}_n \cdot u_n$  ( $\tilde{u}'_n = 0$  on  $\mathbb{D}(n-1)$ ). We have that  $\bar{\partial}\tilde{u}_n = v + \bar{\partial}\tilde{u}'_n$ . Solve  $\bar{\partial}u' = \bar{\partial}\tilde{u}'_n$  according to Proposition 4.4. Then  $\bar{\partial}(\tilde{u}_n - u') = v$ . If we let  $\pi$  denote the orthogonal projection  $\pi : L^2(L_t, \sigma_t + s\psi) \rightarrow \mathcal{O}L^2(L_t, \sigma_t + s\psi)^\perp$ , we need to estimate  $\|u_n - \pi(\tilde{u}_n - u')\|_{L^2(L_t, \sigma_t + s\psi)}$ . To simplify notation we denote the norm by  $\|\cdot\|_t$ .

We have

$$\begin{aligned} \|u_n - \pi(\tilde{u}_n - u')\|_t &= \|\chi_n \cdot u_n + (1 - \chi_n) \cdot u_n - \pi(\tilde{u}_n - u')\|_t \\ &\leq \|\tilde{u}_n - \pi(\tilde{u}_n)\|_t + \|(1 - \chi_n) \cdot u_n\|_t + \|u'\|_t \\ &\leq \|\tilde{u}_n - \pi(\tilde{u}_n)\|_t + (1 + \sqrt{c_1})\|u_n\|_{L^2(L_t|_{A_n}, \sigma_t + s\psi)} \end{aligned}$$

Note that

$$\|\tilde{u}_n - \pi(\tilde{u}_n)\| \leq \sup_{\substack{f \in \mathcal{O}L^2_t \\ \|f\|_{L^2_t} \leq 1}} \{|\langle \tilde{u}_n, f \rangle|\}.$$

Let  $f \in \mathcal{O}L^2(L, \sigma_t + s\psi)$  with  $\|f\|_{L^2(L, \sigma_t + s\psi)} = 1$ . We have that  $\langle (\chi_n + (1 - \chi_n)u_n), f \rangle = 0$ , and so  $|\langle \tilde{u}_n, f \rangle| = |\langle (1 - \chi_n)u_n, f \rangle|$ . By the Cauchy-Schwartz inequality we get that

$$|\langle \tilde{u}_n, f \rangle| \leq \|u_n\|_{L^2(L_t|_{A_n}, \sigma_t + s\psi)}.$$

□

**Lemma 4.7.** *Let  $U \subset \mathbb{C}$  be a domain, let  $L \rightarrow U$  be a line bundle, and let  $c > 0$ . Then for any  $\epsilon > 0$  there exists a  $\delta > 0$  such that the following holds: Let  $\sigma_1$  and  $\sigma_2$  be metrics on  $L$  with  $dd^c\sigma_j \geq c \cdot dV$ , and assume that  $\|\sigma_1 - \sigma_2\|_{U,1} \leq \delta$ . Let  $v_i \in L^2_{0,1}(L, \sigma_i)$ , for  $i = 1, 2$ , and let  $u_i$  denote the  $L^2(L, \sigma_i)$ -minimal solution to the equation  $\bar{\partial}u_i = v_i$  for  $i = 1, 2$ . Then*

$$\|u_1 - u_2\|_{L^2(L, \sigma_1)} \leq c^{-1}\|v_1 - v_2\|_{L^2(L, \sigma_1)} + \epsilon\|v_2\|_{L^2(L, \sigma_2)}.$$

*Proof.* To shorten notation let  $\|\cdot\|_j$  denote the  $L^2$ -norm with respect to the weight  $\sigma_j$  for  $j = 1, 2$ . Let  $\pi_1$  denote the orthogonal projection  $\pi_1 : L^2(L, \sigma_1) \rightarrow \mathcal{OL}^2(L, \sigma_1)^\perp$ . Then  $\pi_1(u_1 - u_2) = u_1 - \pi_1(u_2)$  satisfies

$$\|u_1 - \pi_1(u_2)\|_1 \leq c^{-1} \|v_1 - v_2\|_1.$$

Hence, we need to show that if  $\delta$  is small enough, then  $\|u_2 - \pi_1(u_2)\|_1 \leq \epsilon \|v_2\|_2$ . For this it is enough to show that if  $g \in \mathcal{OL}^2(L, \sigma_1)$ ,  $\|g\|_1 \leq 1$ , then  $|\langle u_2, g \rangle_1| \leq \epsilon \|v_2\|_2$ . We have that

$$|\langle u_2, g \rangle_1| = \left| \int_U \int_U u_2 \cdot \bar{g} e^{-\sigma_1} dV \right| = \left| \int_U \int_U u_2 \cdot \overline{g \cdot e^{\sigma_2 - \sigma_1}} e^{-\sigma_2} dV \right|.$$

Let  $\tilde{v}$  denote the form  $\tilde{v} := g \bar{\partial}(e^{\sigma_2 - \sigma_1})$ . Clearly, for any  $\epsilon_1 > 0$ , we may choose  $\delta > 0$  small enough such that  $\|\tilde{v}\|_2 \leq \epsilon_1$ . Let  $\tilde{u}$  be the  $L^2(L, \sigma_2)$ -minimal solution to  $\bar{\partial}\tilde{u} = \tilde{v}$ . We have that

$$\|\tilde{u}\|_2 \leq c^{-1} \cdot \epsilon_1.$$

We have that

$$\int_U \int_U u_2 \cdot \overline{g \cdot e^{\sigma_2 - \sigma_1} - \tilde{u}} \cdot e^{-\sigma_2} dV = 0,$$

and so it is enough to estimate  $|\langle u_2, \tilde{u} \rangle_2|$ , and by Cauchy-Schwarz we have that

$$|\langle u_2, \tilde{u} \rangle_2| \leq \|u_2\|_2 \cdot c^{-1} \cdot \epsilon_1 \leq c^{-2} \cdot \epsilon_1 \cdot \|v_2\|_2.$$

□

*Proof of Theorem 4.3:*

We may assume that  $L_{\mathbb{T}}$  is the trivial bundle over  $\mathbb{D} \times \mathbb{T}$  (solve Cousin II using the Cauchy integral formula for solving  $\bar{\partial}$ .)

As stated, for each  $t$  let  $u_t$  denote the  $L^2(\sigma_t + s\psi)$ -minimal solution to the equation  $\bar{\partial}u_t = v_t$ . Let  $\epsilon > 0$ . For each  $n \in \mathbb{N}$  and  $t \in \mathbb{T}$  let  $u_{t,n}$  be the  $L^2(\sigma_t + s\psi|_{\mathbb{D}(n)})$ -minimal solution to the equation  $\bar{\partial}u_{t,n} = v_t|_{\mathbb{D}(n)}$ . To simplify notation we denote the norms by  $\|\cdot\|_t$  and  $\|\cdot\|_{t,n}$ . Fix  $t_0 \in \mathbb{T}$ . Then  $u_{t_0,n}$  converges to  $u_{t_0}$  in  $L^2(\sigma_{t_0} + s\psi)$  and there exists an  $N \in \mathbb{N}$  such that

- i)  $\|u_{t_0} - u_{t_0,n}\|_{t_0,n} \leq \epsilon$ , and
- ii)  $\|u_{t_0,n}\|_{L^2(\sigma_{t_0} + s\psi|_{A_n})} \leq \epsilon$ ,

for all  $n \geq N$ . Fix  $n_0 \geq N$ . By Lemma 4.7 there exists an open neighborhood  $V$  of  $t_0$  such that

- iii)  $\|u_{t,n_0} - u_{t_0,n_0}\|_{t,n_0} \leq \epsilon$ ,

for all  $t \in V$ . By possibly having to choose a smaller  $V$  we may assume that the any weight  $\sigma_t$  is comparable to  $\sigma_{t_0}$  on  $\mathbb{D}(n_0)$ , i.e., we have that

$$\frac{1}{2} e^{-\sigma_t} \leq e^{-\sigma_{t_0}} \leq 2 e^{-\sigma_t}$$

for all  $t \in V$ . We get that

$$\text{iv) } \|u_{t,n_0}\|_{L^2(\sigma_t+s\psi|_{A_{n_0}})} \leq 3 \cdot \epsilon,$$

for all  $t \in V$ . By Corollary 4.6 we have that

$$\text{v) } \|u_t - u_{t,n_0}\|_{t,n_0} \leq c_2 \cdot 3 \cdot \epsilon,$$

for all  $t \in V$ . We get

$$\begin{aligned} \|u_t - u_{t_0}\|_{t_0,n_0} &\leq 2 \cdot \|u_t - u_{t,n_0}\|_{t,n_0} \\ &\quad + 2 \cdot \|u_{t,n_0} - u_{t_0,n_0}\|_{t,n_0} \\ &\quad + \|u_{t_0,n_0} - u_{t_0}\|_{t_0,n_0} \\ &\leq (6 \cdot c_2 + 3) \cdot \epsilon. \end{aligned}$$

## 5. $\bar{\partial}_b$ ON SUSPENSIONS

Our goal in this section is to prove Theorem 1.1 in the special case that the lamination is a so-called suspension (see below for the construction). For applications we will show that we also get transversal smoothness, and we will allow singular metrics.

**Theorem 5.1.** *Let  $X$  be a compact Riemann surface of genus greater than or equal to two, and assume that we are given a  $\mathcal{C}^1$ -smooth suspension  $g : Y \rightarrow X$ . Then there exists a constant  $s > 0$  such that the following holds:*

*Assume that we are given a line bundle  $L_* \rightarrow X$  with a (possibly singular) metric  $\sigma_*$ . We let  $L = g^*L_*$ ,  $\sigma = g^*\sigma_*$ , and we let  $\tilde{L}$  denote the bundle  $\pi^*L$  and  $\tilde{\sigma}$  denote the metric  $\pi^*\sigma$ . Let  $\psi(\zeta) := \log(1 - |\zeta|^2)$ , and assume that  $dd^c(\tilde{\sigma} + s\psi)$  is positive. Then for any  $\mathcal{C}^1$ -smooth  $(0,1)$ -form  $v$  on  $Y$  with coefficients in  $L$  and  $v \in L_{loc}^2(\sigma)$ , there exists a  $\mathcal{C}^1$ -smooth section  $u \in \Gamma(L)$  with  $u \in L_{loc}^2(\sigma)$  and  $\bar{\partial}_b u = v$ . To obtain transversally continuous solutions it is enough to assume  $v$  is continuous and that  $s = 5$ .*

**Remark 5.2.** A section/form being in  $L_{loc}^2$  means that it is locally integrable in the leaf-direction for each leaf.

**5.1. The construction of suspensions.** Let  $X$  be a compact Riemann surface of genus greater than or equal to two (resp. one), let  $f : \mathbb{D} \rightarrow X$  (resp.  $f : \mathbb{C} \rightarrow X$ ) be a universal covering map, and let  $\Gamma$  be the corresponding Deck-group. Let  $\mathbb{T}$  be a compact smooth manifold, and assume that we are given a homomorphism  $\phi : \Gamma \rightarrow \text{Diff}(\mathbb{T})$ . We let  $\tilde{\Gamma}$  denote the group of diffeomorphisms of  $\mathbb{D} \times \mathbb{T}$  (resp.  $\mathbb{C} \times \mathbb{T}$ ) consisting of elements  $\tilde{\varphi} := (\varphi, \phi(\varphi))$  for  $\varphi \in \Gamma$ , we consider the quotient  $Y := (\mathbb{D} \times \mathbb{T})/\tilde{\Gamma}$  (resp.  $Y := (\mathbb{C} \times \mathbb{T})/\tilde{\Gamma}$ ), and denote the quotient map by  $\pi : \mathbb{D} \times \mathbb{T} \rightarrow Y$  (resp.  $\pi : \mathbb{C} \times \mathbb{T} \rightarrow Y$ ).

For genus  $g_X \geq 2$ , coordinate charts on  $Y$  are given as follows: for a point  $(\zeta, t) \in \mathbb{D} \times \mathbb{T}$  let  $U \subset \mathbb{D}$  be a domain such that  $\varphi(U) \cap U \neq \emptyset$ ,  $\varphi \in \Gamma \Rightarrow \varphi = \text{id}$ . Let  $\tilde{U} := \{[(\zeta, t)] : \zeta \in U, t \in \mathbb{T}\}$  and let  $\Phi_{\tilde{U}} : \tilde{U} \rightarrow U \times \mathbb{T}$  be defined by  $[(\zeta, t)] \mapsto (\zeta, t)$ . Let  $\tilde{V}$  be another chart with  $\tilde{U} \cap \tilde{V} \neq \emptyset$ . Then there is a point  $(\zeta_1, t_1) \in U \times \mathbb{T}$  and a point  $(\zeta_2, t_2) \in V \times \mathbb{T}$  such that

$[(\zeta_1, t_1)] = [(\zeta_2, t_2)]$ , *i.e.*, there is an element  $\varphi \in \Gamma$  such that  $\zeta_2 = \varphi(\zeta_1)$  and  $t_2 = \phi(\varphi)(t_1)$ . So the transition  $\Phi_{\tilde{V}, \tilde{U}}$  between  $\Phi_{\tilde{U}}(\tilde{U} \cap \tilde{V})$  and  $\Phi_{\tilde{V}}(\tilde{U} \times \tilde{V})$  is given by  $(\zeta, t) \mapsto (\varphi(\zeta), \phi(\varphi)(t))$ . This gives  $Y$  the structure of a Riemann surface lamination, and the leaves are the images  $\pi(\mathbb{D} \times \{t\})$ ,  $t \in \mathbb{T}$ . There is a natural projection  $g : Y \rightarrow X$ , given by  $[(\zeta, t)] \mapsto [\zeta]$ , and each fiber  $Y_x := g^{-1}(x)$  is diffeomorphic to  $\mathbb{T}$ . The lamination  $Y$  is called a *suspension* over  $X$ .

Now we want to define a transversal metric on  $Y$  and describe a relationship with the Poincaré metric  $d_P$  on  $X$ . Let  $\{U_j\}_{j=1}^m$  be a cover of  $X$  by smoothly bounded disks. We have charts

$$\Phi_j : g^{-1}(U_j) \rightarrow U_j \times \mathbb{T},$$

respecting the projection to  $U_j$ . Let  $d_{\mathbb{T}}$  be any smooth Riemannian distance on  $\mathbb{T}$ , and for each  $j$  let  $d_j$  denote the transversal metric  $d_j := \Phi_j^* d$ . Note that any two distances  $d_i$  and  $d_j$  are comparable on a common domain of definition.

Let  $\{\psi_j\}_{j=1}^m$  be a partition of unity with respect to the given cover of  $X$ , and define a global transversal distance  $d_x(t_1, t_2)$  by

$$d_x(t_1, t_2) := \sum_j \psi_j(x) d_j(t_1, t_2).$$

For each  $j$  and for each  $i$  let  $d_{ij} = (\Phi_j)_* d_i$ . Then on  $U_j \times \mathbb{T}$  we have that  $(\Phi_j)_* d$  is given by

$$[(\Phi_j)_* d]_x(t_1, t_2) = \sum_i \psi_i(x) \cdot d_{ij}(t_1, t_2).$$

For each  $j$ , let  $C_j > 0$  be a constant such that the following holds: if  $x$  and  $y$  are points in  $U_j$ ,  $t_1, t_2 \in \mathbb{T}$ , and  $\gamma$  is a smooth curve connecting  $x$  and  $y$ , then

$$[(\Phi_j)_* d]_y(t_1, t_2) \leq C_j^{l_P(\gamma)} \cdot [(\Phi_j)_* d]_x(t_1, t_2),$$

where  $l_P$  denote the Poincaré length. Choosing a constant  $C$  which is greater than  $C_j$  for all  $j$  we obtain:

**Lemma 5.3.** *Given a  $\mathcal{C}^1$ -smooth suspension  $g : Y \rightarrow X$ , and a transversal metric  $d_x$  as described above, there exists a constant  $C > 0$  such that the following holds:*

*Let  $x \in X$ , let  $t_1^x, t_2^x \in \mathbb{T}_x$ , and let  $\gamma : [0, 1] \rightarrow X$  be a smooth immersion with  $\gamma(0) = x$ . Let  $y = \gamma(1)$ , and for  $j = 1, 2$  let  $t_j^y \in \mathbb{T}_y$  be the point obtained by lifting  $\gamma$  to the leaf  $\mathcal{L}_{t_j^x}$  with initial point  $t_j^x$ , *i.e.*,  $t_j^y$  is its end point. Then*

$$d_y(t_1^y, t_2^y) \leq C^{l_P(\gamma)} \cdot d_x(t_1^x, t_2^x).$$

**Lemma 5.4.** *Given a  $C^1$ -smooth suspension  $g : Y \rightarrow X$  there exists a constant  $k \in \mathbb{N}$  such that the following holds:*

*If  $\varphi \in \Gamma$  satisfies  $\varphi(0) \in A_n$  then*

$$d_0(\phi(\varphi)^{-1}(t_1^0), \phi(\varphi)^{-1}(t_2^0)) \leq 2^{kn} \cdot d_0(t_1^0, t_2^0).$$

*for all points  $t_1^0, t_2^0 \in \mathbb{T}$ . (Here  $d_0$  is the transversal metric constructed above lifted to  $\mathbb{D} \times \mathbb{T}$  and restricted to  $\{0\} \times \mathbb{T} =: \mathbb{T}_0$ .)*

*Proof.* Choose  $k$  such that  $2^k \geq C$  from the previous lemma. Write  $y = \varphi(0)$ . The points  $t_1^y = \phi(\varphi)^{-1}(t_1^0)$  and  $t_2^y = \phi(\varphi)^{-1}(t_2^0)$  are the points that are identified with  $t_1^0$  and  $t_2^0$  respectively by the map  $\phi(\varphi)$ , i.e.,  $(y, t_1^0) \sim (0, t_1^y)$  and  $(y, t_2^0) \sim (0, t_2^y)$ . One way to locate the points  $t_j^y$  is then to project the points  $(y, t_j^0)$  to  $Y$  by  $\pi$  and the lift them back to  $\mathbb{T}_0$ .

Let  $\tilde{\gamma} : [0, 1] \rightarrow \mathbb{D}$  parametrize the straight line segment between 0 and  $y$ , and let  $\tilde{\gamma}_1(t) = (\tilde{\gamma}(t), t_1^0)$ ,  $\tilde{\gamma}_2(t) = (\tilde{\gamma}(t), t_2^0)$ . The Poincaré length of  $\tilde{\gamma}$  is less than  $2^{n+2}$ . Projecting these curves to  $Y$  and using the previous lemma we see that the transversal distance between the two points  $\pi((y, t_j^0))$  is less than  $2^{k(n+2)} \cdot d_0(t_1^0, t_2^0)$ .  $\square$

**5.2. Proof of Theorem 5.1.** We will first prove that we obtain transversally continuous solutions to the equation  $\bar{\partial}_b u = v$  under the assumption that  $v$  is transversally continuous and  $dd^c(\tilde{\sigma} + s\psi)$  is positive for  $s \geq 5$ .

**5.3. Continuous solutions.** Let  $\mathcal{U} := \{U_{j,*}\}_{j=1}^m$  be a cover of  $X$  by simply connected open sets, and let  $\{\alpha_j\}_{j=1}^m$  be a partition of unity with respect to  $\mathcal{U}$ . Writing  $v = \sum_j v_j := \sum_j (\alpha_j \circ g) \cdot v$  we have reduced to solving  $\bar{\partial}_b u_j = v_j$  for each  $j$ . We focus on such a  $v_j$  and drop the subscript  $j$ .

Let  $\tilde{v}$  denote the form  $\tilde{v} := \pi^* v$  with coefficients in  $\tilde{L}$ . For a fixed  $t \in \mathbb{T}$  we let  $\tilde{v}_t$  denote the  $(0, 1)$ -form  $\tilde{v}_t := v(\cdot, t)$  on the leaf  $\mathbb{D} \times \{t\}$ . The form satisfies

$$\text{a. } \tilde{v}_t = \varphi^* \tilde{v}_{\phi(\varphi)(t)}$$

for all  $\varphi \in \Gamma$  and all  $t \in \mathbb{T}$ . We will find transversally continuous solutions  $\tilde{u}_t$  to the equations  $\bar{\partial}_b \tilde{u}_t = \tilde{v}_t$  such that

$$\text{b. } \tilde{u}_t = \varphi^* \tilde{u}_{\phi(\varphi)(t)}$$

for all  $\varphi \in \Gamma$  and all  $t \in \mathbb{T}$ . We get that  $u := \pi_* \tilde{u}$  is well defined and solves  $\bar{\partial}_b u = v$ .

Let  $U_{id}$  be one of the pre-images  $f^* U_*$ . For simplicity of notation we assume that  $0 \in U_{id}$ . For any  $\varphi \in \Gamma$  we let  $U_\varphi := \varphi(U_{id})$ . We write  $\tilde{v}_\varphi := \tilde{v}|_{U_\varphi \times \mathbb{T}}$ . Then  $\tilde{v} = \sum_\varphi \tilde{v}_\varphi$ . We will solve  $\bar{\partial}_b \tilde{u}_\varphi = \tilde{v}_\varphi$  for each  $\varphi$  and then define  $\tilde{u} := \sum_\varphi \tilde{u}_\varphi$ .

For a fixed  $\varphi$  and a fixed  $t$  we do this as follows. Let  $\tilde{v}_{\varphi,t}^* := \varphi^* \tilde{v}_{\varphi,t}$ , and let  $\tilde{u}_{\varphi,t}^*$  be the  $L^2(\tilde{\sigma} + s\psi)$ -minimal solution to the equation  $\bar{\partial}u = \tilde{u}_{\varphi,t}^*$  (note that  $L$  is  $\Gamma$ -invariant). Define  $\tilde{u}_{\varphi,t} = \varphi_* \tilde{u}_{\varphi,t}^*$ . We need to check that

1. the sum  $\tilde{u}_t := \sum_{\varphi} \tilde{u}_{\varphi,t}$  converges for each fixed  $t$ ,
2. the solutions vary continuously with  $t$ , and
3. the solutions satisfy  $\tilde{u}_t = \varphi^* \tilde{u}_{\phi(\varphi)(t)}$ .

To show 1. it is enough to show that the sum converges in  $L^2(\tilde{\sigma})$  for each fixed  $t$ . Let  $c_1$  be a constant such that  $\|\tilde{v}_{\varphi,t}^*\|_{L^2(\tilde{L}, \tilde{\sigma} + s\psi)} \leq c_1$  for all  $\varphi$  and all  $t$ . According to Hörmander there exists a constant  $c_2$  such that  $\|\tilde{u}_{\varphi,t}^*\|_{L^2(\tilde{L}, \tilde{\sigma} + s\psi)} \leq c_1 \cdot c_2$  for all  $\varphi$  and all  $t$ . Fix  $0 < r < 1$ . According to Lemma 3.8 there exists a constant  $c_3$  such that if  $\varphi^{-1}(0) \in A_n$ , then  $e^{-\psi(\zeta)} \geq c_3 \cdot 2^n$  for all  $\zeta \in \varphi^{-1}(\mathbb{D}_r)$ . According to Lemma 3.9 there exists a constant  $c_4$  such that the number of  $\varphi$ -s such that  $\varphi^{-1}(0) \in A_n$  is no more than  $c_4 \cdot 2^n$ . According to Lemma 3.5 we have that  $|\varphi'(\zeta)| \leq 2^{n+2}$  for all  $\zeta \in \mathbb{D}$  if  $\varphi(0) \in A_n$ .

We get that

$$\begin{aligned}
\sum_{\varphi} \|\tilde{u}_{\varphi,t}\|_{L^2(L|\mathbb{D}_r, \tilde{\sigma})} &= \sum_{\varphi} \sqrt{\int \int_{\mathbb{D}_r} |\tilde{u}_{\varphi,t}|^2 e^{-\tilde{\sigma}} dV} \\
&= \sum_{\varphi} \sqrt{\int \int_{\varphi^{-1}(\mathbb{D}_r)} \varphi^* [|\tilde{u}_{\varphi,t}|^2 e^{-\tilde{\sigma}} dV]} \\
&= \sum_n \sum_{\varphi^{-1}(0) \in A_n} \sqrt{\int \int_{\varphi^{-1}(\mathbb{D}_r)} \varphi^* [|\tilde{u}_{\varphi,t}|^2 e^{-\tilde{\sigma}} dV]} \\
&\leq \sum_n \sum_{\varphi^{-1}(0) \in A_n} 2^{n+2} \cdot \sqrt{\int \int_{\varphi^{-1}(\mathbb{D}_r)} |\tilde{u}_{\varphi,t}^*|^2 e^{-\tilde{\sigma}} \cdot dV} \\
&\leq \sum_n \sum_{\varphi^{-1}(0) \in A_n} 4c_3^{-s/2} \cdot \left(\frac{1}{2}\right)^{n(s-2)/2} \sqrt{\int \int_{\varphi^{-1}(\mathbb{D}_r)} |\tilde{u}_{\varphi,t}^*|^2 e^{-\tilde{\sigma} - s\psi} dV} \\
&\leq 4c_1 c_2 c_4 c_3^{-s/2} \cdot \sum_n \left(\frac{1}{2}\right)^{n(s-4)/2}.
\end{aligned}$$

This concludes the proof of 1.

To show 2. fix  $t_1 \in \mathbb{T}$  and  $\epsilon > 0$ . Fix any integer  $N \in \mathbb{N}$  such that  $8c_1 c_2 c_4 c_3^{-s/2} \cdot \sum_{n \geq N} \left(\frac{1}{2}\right)^{n(s-4)/2} < \frac{\epsilon}{2}$ . For any  $\delta > 0$  we get, by the transversal continuity of  $v$ , that for all  $t_2$  close enough to  $t_1$  we have

$$\|\tilde{v}_{\varphi,t_1}^* - \tilde{v}_{\varphi,t_2}^*\|_{L^2(\tilde{L}, \tilde{\sigma} + s\psi)} \leq \delta \text{ for all } \varphi \text{ with } \varphi^{-1}(0) \in A_n, n \leq N$$



and so by Hörmander we get that

$$\|\tilde{u}_{\varphi,t_1}^* - \tilde{u}_{\varphi,t_2}^*\|_{L^2(\tilde{L}, \tilde{\sigma} + s\psi)} \leq \delta c_2 \text{ for all } \varphi \text{ with } \varphi^{-1}(0) \in A_n, n \leq N.$$

By a calculation similar to that above we see that

$$\sum_{\varphi} \|\tilde{u}_{\varphi,t_1} - \tilde{u}_{\varphi,t_2}\|_{L^2(L|_{\mathbb{D}_r}, \tilde{\sigma})} \leq 4\delta c_2 c_4 c_3^{-s/2} \cdot \sum_n \left(\frac{1}{2}\right)^{n(s-4)/2} + \frac{\epsilon}{2},$$

hence the solutions vary continuously with  $t$ .

To show 3. note first that  $a$ . amounts to saying that

$$\tilde{v}_{\varphi,t} = \tau^* \tilde{v}_{(\tau \circ \varphi), \phi(\tau)(t)},$$

for all  $\varphi, \tau \in \Gamma$  and  $t \in \mathbb{T}$ , and consequently

$$\varphi^* \tilde{v}_{\varphi,t} = (\tau \circ \varphi)^* \tilde{v}_{(\tau \circ \varphi), \phi(\tau)(t)}.$$

It follows that

$$\tilde{u}_{\varphi,t} = \varphi_* \tilde{u}_{\varphi,t}^* = \tau^* [(\tilde{\tau} \circ \varphi)_* \tilde{u}_{(\tau \circ \varphi), \phi(\tau)(t)}^*] = \tau^* \tilde{u}_{(\tau \circ \varphi), \phi(\tau)(t)},$$

for all  $\varphi, \tau \in \Gamma$  which is equivalent to 3.

The proof that we obtain transversally continuous solutions with  $s \geq 5$  is complete, and we proceed to show that the solutions are transversally smooth if  $s$  is large enough.

#### 5.4. A Lipschitz estimate.

**Lemma 5.5.** *Fix  $0 < r < 1$ . Then there exists a constant  $c > 0$  such that if  $\varphi \in \Gamma$  satisfies  $\varphi(0) \in A_n$ , and  $t_1, t_2 \in \mathbb{T}_0$ , then*

$$\|u_{\varphi,t_2} - u_{\varphi,t_1}\|_{L^2(\tilde{L}|_{\mathbb{D}_r}, \tilde{\sigma})} \leq c \cdot \left(\frac{1}{2}\right)^{n(s-2(k+1))/2} \cdot d_0(t_1, t_2).$$

*Proof.* Note first that by the assumption that the family  $v_{id,t}$  is smooth, and  $\mathbb{T}$  is compact, there exists a constant  $c_1 > 0$  such that

$$1) \|v_{id,t'_1} - v_{id,t'_2}\|_{U_{id}} \leq c_1 \cdot d_0(t'_1, t'_2),$$

for all  $t'_1, t'_2 \in \mathbb{T}$  (note that we are taking the sup-norm). By possibly having to increase  $c_1$  depending on  $s$ , we get the corresponding  $L^2$ -estimate

$$2) \|v_{id,t'_1} - v_{id,t'_2}\|_{L^2(\tilde{L}, \tilde{\sigma} + s\psi)} \leq c_1 \cdot d_0(t'_1, t'_2).$$

Let  $v_{id,t'_1} = \varphi^* v_{\varphi,t_1}$  and  $v_{id,t'_2} = \varphi^* v_{\varphi,t_2}$ . By Lemma 5.4 we have that  $d_0(t'_1, t'_2) \leq 2^{kn} \cdot d_0(t_1, t_2)$ , and so by 2) we get that

$$2') \|\varphi^* v_{\varphi,t_1} - \varphi^* v_{\varphi,t_2}\|_{L^2(\tilde{L}, \tilde{\sigma} + s\psi)} \leq c_1 \cdot 2^{kn} \cdot d_0(t_1, t_2).$$

Since  $u_{\varphi,t_1}^* - u_{\varphi,t_2}^*$  is the  $L^2(\tilde{\sigma} + s\psi)$ -minimal solution to the equation  $\bar{\partial}u = \varphi^* v_{\varphi,t_1} - \varphi^* v_{\varphi,t_2}$  we get that

$$3) \|u_{\varphi,t_1}^* - u_{\varphi,t_2}^*\|_{L^2(\tilde{L}, \tilde{\sigma} + s\psi)} \leq c_1 c_2 \cdot 2^{kn} \cdot d_0(t_1, t_2).$$

A calculation similar to that above gives that

$$\|u_{\varphi,t_2} - u_{\varphi,t_1}\|_{L^2(\tilde{L}|_{D_r}, \tilde{\sigma})} \leq c \cdot \left(\frac{1}{2}\right)^{n(s-2(k+1))/2} \cdot d_0(t_1, t_2).$$

□

### 5.5. Smoothness term by term.

**Lemma 5.6.** *Let  $U \subset \subset \mathbb{D}$  and let  $\tilde{v}_t$  be a smooth family of smooth  $(0,1)$ -forms with coefficients in  $\tilde{L}$ , each one supported in  $U$ . For each  $t \in \mathbb{T}$  let  $\tilde{u}_t$  be the  $L^2(\tilde{\sigma} + s\psi)$ -minimal solution to the equation  $\bar{\partial}u = \tilde{v}_t$ . Then  $\tilde{u}_t$  is a smooth family of sections of  $\tilde{L}$ .*

*Proof.* Let  $t_0 \in \mathbb{T}$  and let  $\gamma_t$  be a continuous vector field on a neighborhood  $\Omega$  of  $t_0$ . For each  $t \in \Omega$  let  $\tilde{v}_t^{\gamma_t}$  be the  $(0,1)$ -form obtained by differentiating  $\tilde{v}_t$  with respect to  $\gamma_t$ . Then  $\tilde{v}_t^{\gamma_t}$  is a continuous family of  $(0,1)$ -forms. Note that if  $\gamma_{t,s}$  is a continuous family of vector fields parametrized by  $s$ , then  $\tilde{v}_t^{\gamma_{t,s}}$  is continuous also in  $s$ .

For each  $t$  let  $\tilde{u}_t^{\gamma_t}$  be the  $L^2(\tilde{\sigma} + s\psi)$ -minimal solution to the equation  $\bar{\partial}u = \tilde{v}_t^{\gamma_t}$ . Then  $\tilde{u}_t^{\gamma_t}$  is a continuous family of sections. We claim that at any point  $(\zeta, t)$  we have that  $\gamma_t(\tilde{u}_t)(\zeta) = \tilde{u}_t^{\gamma_t}(\zeta)$ .

Let  $\gamma : (-1, 1) \rightarrow \mathbb{T}$  be a smooth curve with  $\gamma(0) = t$  and  $\gamma'(0) = \gamma_t$ . Let  $\delta\gamma$  denote the point  $\gamma(\delta)$ . Clearly

$$\bar{\partial}\left(\frac{\tilde{u}_{t+\delta\gamma} - \tilde{u}_t}{\delta}\right) = \frac{\tilde{v}_{t+\delta\gamma} - \tilde{v}_t}{\delta},$$

and it is also the  $L^2(\tilde{\sigma} + s\psi)$ -minimal solution. Since the right-hand side converges uniformly it follows by Hörmanders estimate that

$$\frac{\tilde{u}_{t+\delta\gamma} - \tilde{u}_t}{\delta}$$

converges uniformly to  $\tilde{u}_t^{\gamma_t}$ .

Finally we could check the continuity of the disk-derivatives by the same method, or we could produce another solution whose disk-derivatives vary continuously using the Cauchy-formula, and then conclude by the Cauchy-estimates.

□

**5.6. Smoothness of the sum.** Fix  $t \in \mathbb{T}$ . We will show that  $u$  is smooth near  $\mathbb{D} \times \{t\}$ . Using local coordinates on  $\mathbb{T}$  we may assume that the point  $t$  is the origin in  $\mathbb{R}^n$  and that the metric is the usual one (since everything is comparable). Let  $\gamma$  be a vector of norm one in  $\mathbb{R}^n$ . We need to estimate

$$\left\| \frac{\tilde{u}_{t+\delta\gamma} - \tilde{u}_t}{\delta} \right\|_{L^2(\tilde{L}|_{\mathbb{D}_r}, \tilde{\sigma})},$$

where  $\tilde{u}_t$  is defined by 1. above.

Using Lemma 5.5 and following the arguments for continuity we see that

$$\begin{aligned} \left\| \frac{\tilde{u}_{t+\delta\gamma} - \tilde{u}_t}{\delta} \right\|_{L^2(\tilde{L}|_{\mathbb{D}_r}, \tilde{\sigma})} &\leq \sum_{\varphi^{-1}(0) \in A_n, n \leq N} \left\| \frac{\tilde{u}_{\varphi, t+\delta\gamma} - \tilde{u}_{\varphi, t}}{\delta} \right\|_{L^2(\tilde{L}|_{\mathbb{D}_r}, \tilde{\sigma})} \\ &\quad + \sum_{n > N} c \cdot \left(\frac{1}{2}\right)^{n(s-2(k+2))/2}, \end{aligned}$$

for any  $N \in \mathbb{N}$ . Note also that all but a finite number of  $(\tilde{u}_{t+\delta\gamma} - \tilde{u}_t)$ -s are holomorphic on  $\mathbb{D}_r$ . Smoothness follows by Lemma 5.6, and the proof of Theorem 5.1 is complete.  $\square$

**Remark 5.7.** It is also possible to solve  $\bar{\partial}_b$  on suspensions over tori. In that case it can be done more explicitly by following the above procedure, but using a weighted Cauchy integral formula for solving  $\bar{\partial}$ :

$$u(z) := \frac{-1}{\pi \cdot z^k} \int_{\mathbb{C}} \int \frac{v(\zeta) \zeta^k}{\zeta - z} d\zeta \wedge d\bar{\zeta},$$

on the universal cover  $\mathbb{C} \rightarrow \mathbb{C}/\Gamma$ .

## 6. PROOF OF THEOREM 1.2

In [12] we proved with Sibony that there exists a smooth suspension over a compact Riemann surface of genus two, which is a minimal lamination supporting uncountably many extremal closed laminated currents which are mutually singular. The lamination is of real transverse dimension two. It therefore suffices to prove the following:

**Theorem 6.1.** *Let  $g : Y \rightarrow X$  be a  $\mathcal{C}^1$ -smooth suspension over a compact Riemann surface  $X$  of genus  $g_X \geq 2$ . Then  $Y$  is projective.*

*Proof.* We need to find a line bundle  $L \rightarrow Y$  where we can find enough sections to separate points and to have non-vanishing differentials. We will do this by constructing local sections and solving  $\bar{\partial}_b$  with singular weights.

Fix  $s$  according to Theorem 5.1, and define

$$\tilde{\omega}(\zeta) := \frac{1}{(1 - |\zeta|^2)^2} dV, \zeta \in \mathbb{D}.$$

Since  $\tilde{\omega}$  is  $\Gamma$ -invariant it defines a volume form  $\omega_* := f_* \tilde{\omega}$  on  $X$ .

Note that there exists a constant  $c_1 > 0$  such that the following holds

- a. for every pair  $x_1, x_2 \in X$  there exists a function  $\tau_* \in L_{loc}^2(X)$  with  $\nu(\tau_*, x_j) = 2$  for  $j = 1, 2$ , and  $dd^c \tau_* \geq -c_1 \cdot \omega_*$ .

Here  $\nu$  denotes the Lelong number. The claim follows by compactness of  $X$  and the construction of such a  $\tau$  for fixed  $x_1, x_2$  (see 2.2.1 of [12] for details).

Since  $X$  is projective there exists a line bundle  $L_* \rightarrow X$  with a smooth metric  $\sigma_*$  such that

b.  $dd^c \sigma_* \geq c_2 \cdot \omega_*$ , for some  $c_2 > 0$ .

At this point we fix  $N \in \mathbb{N}$  such that

c.  $N \cdot c_2 > c_1 + 4s$ .

Define  $\tilde{\sigma} := f^* \sigma_*$  and  $\sigma := g^* \sigma_*$ , and for any such  $\tau_*$  we define  $\tilde{\tau} := f^* \tau_*$  and  $\tau := g^* \tau_*$ . By a. and b. we have that

$$dd^c(k \cdot \tilde{\sigma} + \tilde{\tau} + s\psi) \geq (k \cdot c_2 - c_1 - 4s) \cdot \tilde{\omega},$$

and so by c. we have that for all  $k \geq N$  we may solve  $\bar{\partial}_b$  for  $L^2$ -sections of  $L^{\otimes k}$  over  $Y$ , and the solutions will be in  $L^2_{loc}(k\sigma + \tau)$  as well as being  $\mathcal{C}^1$ -smooth on the total space. The main point is that this will force the solutions to vanish to order two in the leaf direction along the transversals over the points  $x_1$  and  $x_2$ . We now sketch the steps to produce sufficiently many sections of  $L^{\otimes k}$  to produce an embedding.

- i. *Non-vanishing differentials in the leaf direction:* Here we can use sections of the bundle  $L^{\otimes k}_*$ . Given a point  $x \in X$  let  $\xi_1$  be a smooth section of  $L^{\otimes k}_*$ , holomorphic near  $x$ , which in local coordinates ( $x = 0$ ) looks like  $\xi_1(z) = z + O(|z|^2)$ . Let  $\xi_2$  be a section that looks like  $\xi_2(z) = 1 + O(|z|^2)$ . Let  $v_j := \bar{\partial} \xi_j$ , solve  $\bar{\partial}_b u_j = v_j$  with metric which is singular at  $x$ , and define  $s_j := \xi_j - u_j$ . The quotient  $s_1/s_2$  has a non-vanishing differential in the leaf direction on a full neighborhood of the transversal  $\mathbb{T}_x$ . By compactness we cover all of  $X$ .
- ii. *Separate points over different points in the base:* Use a similar construction as i. to separate transversals  $\mathbb{T}_{x_j}$  and  $\mathbb{T}_{x_i}$  for  $x_i \neq x_j$ . Use compactness and i. to cover everything.
- iii. *The transversal direction:* For any given  $x \in X$  start with smooth sections  $\xi_1, \dots, \xi_m$  of  $L^{\otimes k}|_{\mathbb{T}_x}$  providing an embedding of  $\mathbb{T}_x$  into projective space. Extend the sections  $\xi_j$  constantly along leaves near  $\mathbb{T}_x$  and use a cut-off function on the base to extend each  $\xi_j$  to a section of  $L^{\otimes k}$ . Define  $v_j := \bar{\partial}_b \xi_j$ , solve  $\bar{\partial}_b u_j = v_j$  with a weight which is singular along  $\mathbb{T}_x$ , and define  $s_j := \xi_j - u_j$ . Then each  $s_j$  will have the same differential as  $\xi_j$  along  $\mathbb{T}_x$ , and since they are all  $\mathcal{C}^1$ -smooth, they provide an embedding of all transversals near  $\mathbb{T}_x$ . By compactness we cover all transversals.

Note that it was only for iii. we used Theorem 5.1.

□

## 7. PROOF OF THE MAIN THEOREM

Let

$$\phi_\alpha = (z_\alpha, t_\alpha) : U_\alpha \rightarrow \mathbb{D} \times \mathbb{T}$$

be a flow-box. Let  $\mathbb{T}_0$  denote the transversal  $\phi_\alpha^{-1}(\{0\} \times \mathbb{T})$ . By the continuity of the Kobayashi metric we may choose a constant  $r > 0$  such that

- a. if  $\gamma : [0, 1] \rightarrow \mathcal{L}_t$  is a smooth curve such that,  $\gamma(0), \gamma(1) \in U_\alpha$ ,  $z_\alpha(\gamma(0)) = z_\alpha(\gamma(1)) = 0$  and  $t_\alpha(\gamma(0)) \neq t_\alpha(\gamma(1))$ , then the Kobayashi length of  $\gamma$  is greater than  $r$ .

Simply let  $r$  be the infimum of the Kobayashi radii of the plaques  $\mathcal{L}_{\alpha, t_\alpha(t)}$  for  $t \in \mathbb{T}_0$ . Similarly we may choose  $r$  such that

- b. if  $\gamma : [0, 1] \rightarrow \mathcal{L}_t$  is a non-trivial curve with  $\gamma(0) = \gamma(1) = t \in \mathbb{T}_0$ , then the length of  $\gamma$  is greater than or equal to  $r$ .

Choose  $0 < r' < r$ , and we get that

- c. the K-disk  $\triangle_{K, r'}(t)$  in  $\mathcal{L}_t$  of radius  $r'$  centered at  $t$  is contained in  $\mathcal{L}_{\alpha, t_\alpha(t)}$  for all  $t \in \mathbb{T}_0$ .

By compactness it is enough to solve  $\bar{\partial}_b u = v$  when  $v$  is compactly supported in  $\bigcup_{t \in \mathbb{T}_0} \triangle_{K, r'}(t)$ .

Let  $\tilde{\xi}(t)$  be the vector field  $\phi_\alpha^*(\partial/\partial\zeta|_{\zeta=0})$ , and let  $\xi(t)$  be the corresponding vector field normalized by the Kobayashi metric. For each  $t \in \mathbb{T}_0$  let  $f_t : \mathbb{D} \rightarrow \mathcal{L}_t$  be the universal covering map with  $f_t(0) = t$  and  $f_t'(0) = \xi(t)$ . Let  $L_t$  denote the line bundle  $L_t := f_t^*L$  over  $\mathbb{D}$ . Let  $\sigma_t$  denote the metric  $f_t^*\sigma$ . By continuity of the Kobayashi metric we have that

$$dd^c \sigma_t(\zeta) = \frac{g_t(\zeta)}{(1 - |\zeta|^2)^2} dV,$$

where  $g_t$  is bounded from below  $g_t(\zeta) \geq c > 0$  independently of  $t$  and  $\zeta$ . By passing to a power of  $L$  we may assume that  $g_t(\zeta) \geq c > 20$ , and so  $dd^c(\sigma_t + 5\psi)$  is strictly positive independently of  $t$ , i.e., we may solve  $\bar{\partial}$  with estimates using the metric  $\sigma_t + 5\psi$ . Let  $v_t$  denote the form  $v_t := f_t^*v$ .

For each  $t \in \mathbb{T}$  let  $E_t$  denote the discrete set of points  $E_t := \{f_t^{-1}(\mathbb{T}_0)\}$ , and note that the Kobayashi distance between any two points in  $E_t$  is greater than  $r$ . For each point  $\zeta \in E_t$  let  $U_{t, \zeta}$  denote the connected component of  $f_t^{-1}(\mathcal{L}_{\alpha, t_\alpha(f(\zeta))})$  containing  $\zeta$ . Let  $v_{t, \zeta} := v_t|_{U_{t, \zeta}}$  and note that  $v_{t, \zeta}$  is compactly supported in  $U_{t, \zeta}$ . In fact,  $v_{t, \zeta}$  is compactly supported the K-disk of radius  $r'$  centered at  $\zeta$ .

For each  $n = 0, 1, 2, \dots$ , let  $E_{t, n}$  denote the set

$$E_{t, n} = \{\zeta \in E_t : 1 - (\frac{1}{2})^n \leq |\zeta| < 1 - (\frac{1}{2})^{n+1}\}.$$

By Lemma 3.9 the following holds:

- d. there exists a constant  $c > 0$  such that  $\sharp(E_{t, n}) \leq c \cdot 2^n$  for all  $n$ , and for all  $t$ .

For each  $\zeta \in E_t$  let  $\varphi_{t, \zeta}$  denote the element  $\varphi_{t, \zeta} \in \text{Aut}_{hol} \mathbb{D}$  with  $\varphi_{t, \zeta}(0) = \zeta$  and  $(f_t \circ \varphi_{t, \zeta})'(0) = \xi(f_t(\zeta))$ . Let  $v_{t, \zeta}^*$  denote the form  $\varphi_{t, \zeta}^* v_{t, \zeta}$ , let  $u_{t, \zeta}^*$  denote the  $L^2(\varphi_{t, \zeta}^*(\sigma_t) + 5\psi)$ -minimal solution to the equation  $\bar{\partial} u = v_{t, \zeta}^*$ , and let  $u_{t, \zeta} := (\varphi_{t, \zeta})_* u_{t, \zeta}^*$ . We define

$$(*) \quad u_t := \sum_{\zeta \in E_t} u_{t,\zeta},$$

and then finally

$$(**) \quad u := (f_t)_* \left( \sum_{\zeta \in E_t} u_{t,\zeta} \right) \text{ on } \mathcal{L}_t.$$

We need to check that

- i) the sum  $(*)$  converges for each  $t \in \mathbb{T}_0$ ,
- ii) the push-forward  $(**)$  is well defined, and
- iii) the solutions vary continuously between leaves.

i) and iii) are proved essentially as in the special case of a suspension over a genus  $g$  surface,  $g \geq 2$ , but we need Theorems 2.1 and 4.3. For convergence we need to note that, due to continuity of the Kobayashi metric, there exists a constant  $c_1 > 0$  such that

$$\text{e. } \|v_{t,\zeta}^*\|_{L^2(L, \sigma_t^* + 5\psi)} \leq c_1,$$

for all  $\zeta$  and all  $t$ . A similar calculations as before then gives, for a fixed  $0 < r < 1$ , that,

$$\text{f. } \sum_{\zeta \in E_t} \|u_{t,\zeta}\|_{L^2(L_t|_{\mathbb{D}_r}, \sigma_t)} \leq C \cdot \sum_n \sum_{\zeta \in E_{t,n}} \left(\frac{1}{2}\right)^{n(s-2)/2} \cdot \|v_{t,\zeta}^*\|_{L^2(L, \sigma_t^* + 5\psi)},$$

where the constant  $C$  is independent of  $t$ . Then d. and e. gives convergence.

For continuity note first that, by the same calculation as the one leading to f., we get for any  $N \in \mathbb{N}$  that

$$\text{g. } \sum_{n \geq N, \zeta \in E_{t,n}} \|u_{t,\zeta}\|_{L^2(L_t|_{\mathbb{D}_r}, \sigma_t)} \leq C \cdot \sum_{n \geq N} \sum_{\zeta \in E_{t,n}} \left(\frac{1}{2}\right)^{n(s-2)/2} \cdot \|v_{t,\zeta}^*\|_{L^2(L, \sigma_t^* + 5\psi)}.$$

So using d. and e. it follows that for any  $\epsilon > 0$  there exists an  $N \in \mathbb{N}$  such that

$$\text{h. } \sum_{n \geq N, \zeta \in E_{t,n}} \|u_{t,\zeta}\|_{L^2(L_t|_{\mathbb{D}_r}, \sigma_t)} < \epsilon,$$

for all  $t$ . So proving convergence is reduced to proving that finite sums converge. Since  $f_t \xrightarrow[t \rightarrow t_0]{} f_{t_0}$  u.o.c. in  $\mathbb{D}$  (Theorem 2.1), this is easily reduced to showing that  $u_{t,0} \rightarrow u_{t_0,0}$  as  $t \rightarrow t_0$ . This is the content of Theorem 4.3.

ii) Let  $t_1, t_2 \in \mathbb{T}_0$  both be contained in the same leaf  $\mathcal{L}_t$  (we allow them to be the same point). Let  $\psi$  be any element  $\psi \in \text{Aut}_{hol} \mathbb{D}$  such that  $f_{t_2} = f_{t_1} \circ \psi$ . We need to show that  $u_{t_2} = \psi^* u_{t_1}$ .

Let  $\zeta \in E_{t_2}$  and note that  $v_{t_2,\zeta} = \psi^* v_{t_1,\psi(\zeta)}$ , and that

$\varphi_{t_2,\zeta} = \psi^{-1} \circ \varphi_{t_1,\psi(\zeta)}$ . We have that  $v_{t_2,\zeta}^* = \varphi_{t_2,\zeta}^* v_{t_2,\zeta}$ , and so

$$v_{t_2,\zeta}^* = (\psi^{-1} \circ \varphi_{t_1,\psi(\zeta)})^* (\psi^* v_{t_1,\psi(\zeta)}) = \varphi_{t_1,\psi(\zeta)}^* v_{t_1,\psi(\zeta)} = v_{t_1,\psi(\zeta)}^*.$$

We get

$$u_{t_2,\zeta} = (\varphi_{t_2,\zeta})_* u_{t_1,\psi(\zeta)}^* = \psi_*^{-1} (\varphi_{t_1,\psi(\zeta)})_* u_{t_1,\psi(\zeta)}^* = \psi^* u_{t_1,\psi(\zeta)}.$$

From this we see that  $u_{t_2} = \psi^* u_{t_1}$ . This shows that  $u_t$  is well defined on the quotient and that it is independent of the choice of transversal point.

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